Positivity of Hodge Modules, Slope stability, and Foliations on DM stacks $/\mathbb{C}$ (joint with Sebastian Casalaina-Martin)

I) Positivity

II) Slope stability

III) Foliations

IV) WHY?

I.1) Thm (Hodge decomposition): $X/\mathbb{C}$ sm. proj. var.

$$\underbrace{H^m(X,\mathbb{C})}_{\text{weight}=m} = \bigoplus_{p+q=m} H^{p,q}(X), \quad \overline{H^{p,q}} = H^{q,p}$$

$$\rightsquigarrow (H_{\mathbb{Q}},\ H_{\mathbb{Q}}\otimes_{\mathbb{Q}}\mathbb{C} \xrightarrow{\sim} H,\ F^\bullet H,\ Q,\ w)$$

$X \xrightarrow{f} Y$ sm. proj. of rel. dim n $\rightsquigarrow \underline{R^k f_* \mathbb{Q}_X}$ pol. VHS

$$V = (\underset{\text{bundle}}{\mathcal{V}} \xrightarrow[\text{flat connection}]{\nabla} \mathcal{V}\otimes\Omega^1_Y,\ F^\bullet\mathcal{V},\ V_{\mathbb{Q}}\otimes_{\mathbb{Q}}\mathcal{O}_Y \simeq \mathcal{V},\ Q: V_{\mathbb{Q}} \xrightarrow{\sim} V_{\mathbb{Q}}^\vee,\ w)$$

Griffiths transversality:

$$\nabla: F^p \to F^{p-1}\otimes\Omega^1_Y, \quad \forall p$$

$$\theta_p: \frac{F^p}{F^{p+1}} \to \frac{F^{p-1}}{F^p}\otimes\Omega^1_Y \quad \forall p$$

$$E^p \to E^{p-1}\otimes\Omega^1_Y \quad \mathcal{O}_Y\text{-linear}$$

Higgs field

$$X \xrightarrow{f} Y \quad \frac{F^n R^n f_*\mathbb{Q}_X}{F^{n+1}R^n f_*\mathbb{Q}_X} = E^{n,0} = f_*(\omega_{X/Y})$$

$$E^{n,0}_y = H^0(\omega_{X_y}).$$

Thm (Griffiths) V p.VHS, weight n, on Y

$\forall y\in Y$, $\forall v\in T_{Y,y}$, $\forall \eta\in E^{n,0}_y$,

$$\langle \Theta_{n,0}(v,\bar v)\cdot\eta,\eta\rangle = \|\theta_n(v)\cdot\eta\|^2_h \geq 0.$$

$Q \rightsquigarrow$ Hodge metric $\langle\,,\rangle \rightsquigarrow$ Chern connection $\rightsquigarrow \Theta$

Claim: $K := \ker(\theta_n)$ seminegative

$$\langle \Theta_K(v,\bar v)\cdot\eta,\eta\rangle \leq 0$$

Pf. $\Theta_K = \Theta|_K - B^*\wedge B \leftarrow$ second fundamental form

$$\langle\Theta_K(v,\bar v)\cdot\eta,\eta\rangle \underset{G}{=} \cancel{\|\theta_n(v)\cdot\eta\|^2_h}^{\,0} - \|B(v)\cdot\eta\|^2_h \leq 0. \quad \blacksquare$$
$\eta\in K$

Claim:
$$K^\vee \text{ semipositive} \Longrightarrow \underbrace{K^\vee \text{ nef vbdl.}}_{\mathcal{O}_{\mathbb{P}(K^\vee)}(1)\text{ is nef}} \overset{(*)}{\Longrightarrow} K^\vee \text{ weakly positive}$$

Def: A ~~locally free~~ coherent sheaf $\mathcal{F}$ is weakly positive over $\underset{\text{open}}{U\subset Y}$ if $\forall\alpha\in\mathbb{N}$, some/any ample l. bdl. A on Y, $\exists\beta>0$: $S^{\alpha\beta}(\mathcal{F})\otimes A^\beta$ is gen'd by global sections at all points of U.

Pf of $*$: $\mathcal{F} := K^\vee$, $\mathbb{P}_Y(\mathcal{F}) \xrightarrow{\pi} Y \xrightarrow{q}\cdot$

Fix $\alpha\in\mathbb{N}$. Pick A v.a. on Y.

$$M := \underbrace{\mathcal{O}_{\mathbb{P}_Y(\mathcal{F})}(1)^{\otimes\alpha}}_{\text{rel. ample}}\otimes\underbrace{\pi^*A}_{\text{ample horizontally}} \quad \text{is ample on } \mathbb{P}_Y(\mathcal{F}).$$

nef

$\beta>>0$, $H^i(M^\beta\otimes\pi^*A^{-i}) = 0$, $\forall\, 1\leq i\leq d_Y$

$$H^p(\underbrace{R^q\pi_*(M^\beta\otimes\pi^*A^{-i})}_{R^q\pi_*(L^{\alpha\beta})\otimes A^{\beta-i}}) \Rightarrow \underset{0}{\overset{\shortparallel}{H^{p+q}}}(M^\beta\otimes\pi^*A^{-i})$$

$q\neq0$: vanishes

$q=0$: $S^{\alpha\beta}(\mathcal{F})\otimes A^{\beta-i}$

$$H^p(\underbrace{S^{\alpha\beta}(\mathcal{F})\otimes A^\beta}\otimes A^{-i}) = 0, \quad \forall p>0,\ \forall\, 1\leq i\leq d_Y$$

0-regular, A v. ample $\Rightarrow$

$S^{\alpha\beta}(\mathcal{F})\otimes A^\beta$ is globally generated $\blacksquare$

Rk: Exist examples where $f_*\omega_{X/Y} = (\underbrace{R^n f_*\mathcal{O}_X}_{E^{0,n}})^\vee$ is NOT nef.

Thm (Viehweg '83): $f: X\to Y$ fiber space of sm proj. vars $\Rightarrow f_*(\omega_{X/Y}^{\otimes m})$ are weakly positive $\forall m\geq1$.

I.2) Intro to Hodge Modules

$\mathcal{M}\in HM^p(X,w)$ is a polarisable pure Hodge module of weight w on X "mean"

$$\mathcal{M} = (\underset{\mathcal{D}_X\text{ mod}}{M},\ \underset{\mathcal{O}_X\text{-coherent filtration}}{F_\bullet M},\ \underset{\text{Perverse }\mathbb{Q}\text{-sheaf}}{P_{\mathbb{Q}}},\ \alpha,\ \underset{\text{weight}}{w})$$

$$\alpha: P_{\mathbb{Q}}\otimes_{\mathbb{Q}}\mathbb{C} \xrightarrow{\sim} DR(M)$$

$$DR(M) = [M \xrightarrow{\nabla} M\otimes\Omega^1 \xrightarrow{\nabla} M\otimes\Omega^2 \to \dots \to M\otimes\Omega^{d_X}_X][d_X]$$

Fiberwise, think: $(H^*(X_y),\ F^\bullet H^*(X_y),\ \alpha: H^*(X_y,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C} \overset{\sim}{\to} H^*(X_y,\mathbb{C}))$

Thm (Saito '89) $M\in HM^p(Y,w)$

$$\Rightarrow M = \bigoplus_\alpha IC_{S_\alpha}(L_\alpha),\quad S_\alpha\subset Y \text{ irreducible closed subvars}$$

$L_\alpha \sim U_\alpha \underset{\text{open}}{\subseteq} S_\alpha \quad L_\alpha$ pol. VHS

$IC_{S_\alpha}(L_\alpha)$

Def: $K_p(M) := \ker(\theta_p: gr^F_p M \to gr^F_{p+1}M\otimes\Omega^1_Y)$.

I.3) Thm (PW'16, CMZ'26): If $M\in HM^p(\mathcal{X},w)$ with strict support X, then $K_p(M)^\vee$ is weakly positive, $\forall p$.

II) Slope stability: $X/\mathbb{C}$ sm. proj. var.

Def: $\alpha\in N_1(X)_{\mathbb{R}}$ movable curve if $\alpha\cdot D\geq0$ $\forall$ pseff. D.

α-slope: $\mu_\alpha(\mathcal{E}) = \frac{c_1(\mathcal{E})\cdot\alpha}{rk(\mathcal{E})}$.

Thm (CP'11, HL'10, …, CMZ'26) HN, JH filtrations exist; Tensor products of α-ss sheaves is α-ss.

III) Foliations

$[X/G] \rightsquigarrow [CP'19]$

Thm A: (~~…~~'17, CP'15, CMZ'26) $\mathcal{X}$ sm. proj. var. $\Delta\subseteq X$ reduced SNC. If $\Omega^1_X(\log\Delta)$ contains a subsheaf with big determinant then $K_X+\Delta$ is big

Thm B: (CP'15, CMZ'26) X sm. proj. var, $\mathcal{F}\subseteq T_X$ foliation. If $\exists\alpha$ movable curve class so that every nonzero quotient of $\mathcal{F}$ has positive α-slope then $\mathcal{F}$ is an algebraically integrable foliation:

$\exists X \overset{\pi}{\dashrightarrow} Z \quad \ker(d\pi: T_X\to\pi^*T_Z) = \mathcal{F}$

Thm C (CMZ'26, CP'15). In the above, if $\exists\mathcal{G}\subseteq\Omega^1_X(\log\Delta)$ so that $\det(\mathcal{G})$ is big, then $\forall\,\Omega^1_X(\log\Delta)\twoheadrightarrow Q$, $\det(Q)$ is pseff.

Thm B $\Rightarrow$ Thm C $\Rightarrow$ Thm A:

$\mathcal{G}\subset\Omega^1_X(\log\Delta)$ s.t. $\det(\mathcal{G})$ is big

$$0\to\mathcal{G}\to\Omega^1_X(\log\Delta)\to Q\to0$$

$\det(\mathcal{G})$ big; $\det(Q)$ pseff by Thm C

$$\mathcal{O}(K_X+\Delta) = \underbrace{\det(\mathcal{G})}_{\text{big}}\otimes\underbrace{\det(Q)}_{\text{pseff}} \text{ is big.}$$

Sketch of Thm C: If Q not pseff

$\Rightarrow \exists\alpha: \mu_\alpha(Q)<0 \Rightarrow \mu_\alpha(Q^\vee)>0$

$\xrightarrow[\text{stability}]{\text{Slope}}$ α-ss $\mathcal{F}_\Delta\subset T_X(-\log\Delta)$, $\mathcal{F}\subset T_X$ foliation

Thm B $\Rightarrow \mathcal{F}$ alg int $\Rightarrow X\dashrightarrow Z$.
Reduce pseff. to pseff on the fibers, induct $\blacksquare$

IV) WHY?

I + II + III + PS'16 + WW'18 + further work = Main Thm (CMZ'26)

Thm (PS'16, WW'18, CMZ'26) $f: (Y,D)\to\mathcal{X}$ family of KSBA pairs that is relative SNC outside $\Delta\subseteq X$. Assume $\mathrm{Var}(f) = \dim(X)$. Then, $K_X+\Delta$ is big

KSBA moduli spaces are DM stacks and not schemes.